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SEIFERT-FIBERED SURGERIES WHICH DO NOT ARISE FROM PRIMITIVE/SEIFERT-FIBERED CONSTRUCTIONS

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Dedicated to Cameron McA. Gordon on the occasion of his 60th birthday

Abstract. We construct two infinite families of knots each of which admits a Seifert fibered surgery with none of these surgeries coming from Dean's primitive/Seifert-fibered construction. This disproves a conjecture that all Seifert-fibered surgeries arise from Dean's primitive/Seifert-fibered construction. The $(-3, 3, 5)$-pretzel knot belongs to both of the infinite families.

1. Introduction

Let K be a knot in the 3-sphere S^3. Then we denote by $(K;\gamma)$ the 3-manifold obtained by γ-surgery on K, i.e., by attaching a solid torus to $S^3-\mathrm{int}N(K)$ in such a way that γ bounds a meridian disk of the filling solid torus. Using the preferred meridian-longitude pair of $K \subset S^3$, we parametrize slopes γ of K by $r \in \mathbb{Q} \cup \{\infty\}$; then we also write $(K;r)$ for $(K;\gamma)$.

We begin by recalling Berge's [1] construction, an explicit construction which yields several infinite families of knots each admitting a lens space Dehn surgery.

Let K be a knot contained in a genus two Heegaard surface F for S^3, i.e., $S^3 = H \cup_F H'$, where H and H' denote genus two handlebodies. Suppose that K is nontrivial and that the manifolds $H(K)$ and $H'(K)$ are both solid tori, where $H(K)$ (resp. $H'(K)$) is obtained by attaching a 2-handle to H (resp. H') along K. The isotopy class in $\partial N(K)$ of the curve(s) in $\partial N(K) \cap F$ is called the *surface slope* of K with respect to F. Then by performing Dehn surgery on K along the surface slope γ, we obtain a 3-manifold $(K;\gamma) = H(K) \cup H'(K)$, which is a lens space. It cannot be $S^2 \times S^1$ by [11], nor S^3 by [14]. This construction is called *Berge's construction* or the *primitive/primitive construction* and such a knot K is said to be *primitive/primitive* with respect to F.

In [1] Berge suggested the following. See also [13].

Conjecture 1.1. *If $(K;\gamma)$ is a lens space, then this surgery arises from Berge's construction.*

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Dean [7], [8] made a natural modification to Berge's construction; suppose that K is as before except that $H'(K)$ is now a Seifert fiber space over the disk with two exceptional fibers. Then for the surface slope γ, $(K;\gamma)$ is a Seifert fiber space over S^2 with at most three exceptional fibers or a connected sum of two lens spaces. If K is hyperbolic, then the cabling conjecture [12] states that the latter cannot occur. This construction is called *Dean's construction* or the *primitive/Seifert-fibered construction* and such a knot K is said to be *primitive/Seifert-fibered* with respect to F.

The notion of primitive/Seifert-fibered construction has been slightly generalized by allowing the possibility that $H'(K)$ is a Seifert fiber space over the Möbius band with one exceptional fiber [10], [18]. In the following, we use the term primitive/Seifert-fibered construction (or knot) in this generalized sense.

In analogy with Conjecture 1.1, Dean [7] and Gordon [13] asked:

Question 1.2. *If $(K;\gamma)$ is a Seifert fiber space other than a lens space, then does this surgery arise from a primitive/Seifert-fibered construction?*

Many examples of Seifert-fibered surgeries (see, for example, [4], [5], [9] and [10]) have been constructed using the Montesinos trick ([19], [3]). Recently, in [10], Eudave-Muñoz has shown that all known examples of Seifert fibered surgeries constructed by the Montesinos trick can be explained by Dean's construction. Furthermore, Seifert fibered surgeries on twisted torus knots in [17] can also be explained by such constructions [18].

On the other hand, in the present note we demonstrate the following which answers the question above in the negative. A knot K is *strongly invertible* if there is an orientation preserving involution of S^3 which leaves K invariant and reverses an orientation of K; primitive/Seifert-fibered knots are shown to be strongly invertible.

Theorem 1.3. *There is an infinite family of non-strongly invertible knots each of which admits a Seifert-fibered surgery with none of these surgeries arising from the primitive/Seifert-fibered construction. For example, the $(-3,3,5)$-pretzel knot belongs to the family.*

Very recently Hyung-Jong Song has observed that the 1-surgery of the $(-3,3,3)$-pretzel knot is a Seifert-fibered surgery, but does not arise from the primitive/Seifert-fibered construction. In contrast with our examples, the $(-3,3,3)$-pretzel knot is strongly invertible; but it has cyclic period 2 and tunnel number greater than one like ours.

In his thesis [15], the first author observed that the $(-3,3,5)$-pretzel knot has a small Seifert-fibered surgery by experiments via Weeks' computer program SnapPea. This observation is the starting point of our study.

2. Examples

We shall say that a Seifert fiber space is of *type* $S^2(n_1,n_2,n_3)$ if it has a Seifert fibration over S^2 with three exceptional fibers of indices n_1,n_2 and n_3 $(n_i \geq 2)$.

Example 1. Let $K \cup t_1$ be the two component link of Figure 1.

Here K is the Montesinos knot given by the triple of rational tangles $(1/3,-1/3,-1/5)$, which is often called the $(-3,3,5)$-pretzel knot. (We adopt Bleiler's convention [2] on the parametrization of rational tangles.) Let K_n (n is

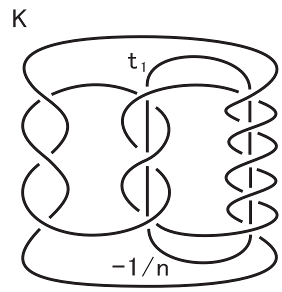

FIGURE 1.

possibly zero) be the knot obtained from K by performing $-1/n$-surgery on t_1. Equivalently, K_n is obtained by doing n-twisting along t_1. Then K_n enjoys the following properties:

(1) K_n is a hyperbolic knot,
(2) K_n has cyclic period 2, but is not strongly invertible,
(3) the tunnel number of K_n is 2, and
(4) $(K_n; 1)$ is a Seifert fiber space of type $S^2(3, 5, |15n + 4|)$.

Before verifying properties (1)–(4) we observe that $\{K_n\}$ is the family of Theorem 1.3.

Proof of Theorem 1.3. Properties (2) and (4) show that K_n is not strongly invertible and admits a Seifert-fibered surgery. Assume for a contradiction that K_n is primitive/Seifert-fibered; then $H(K_n)$ is a solid torus for an unknotted genus 2 handlebody H with $K \subset \partial H$. First we show that K_n has tunnel number 1 following [7]. By [25], there is a homeomorphism of the genus two handlebody H after which K_n appears as in Figure 2. After pushing K_n into H, take an arc t as in Figure 2.

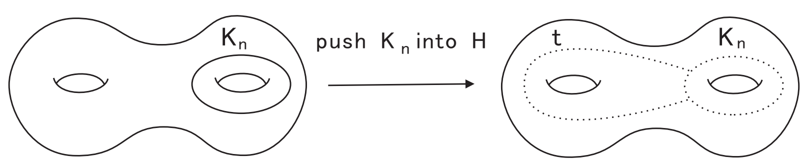

FIGURE 2.

Then $H - \text{int}N(K_n \cup t)$ is the product of a surface and an interval. Thus $S^3 - \text{int}N(K_n \cup t) = H' \cup (H - \text{int}N(K_n \cup t))$ is a genus two handlebody, so the knot K_n has tunnel number 1. This then implies that K_n is strongly invertible by [21, Lemma 5], a contradiction. Hence the Seifert-fibered surgery does not come from the primitive/Seifert-fibered construction. □ (Theorem 1.3)

Claim 2.1. *K_n has cyclic period* 2.

Proof. As shown in Figure 3, let $f : S^3 \to S^3$ be the π-rotation about C such that $f(K) = K$ and $f(t_1) = t_1$. The axis C is disjoint from K and intersects t_1 in exactly two points.

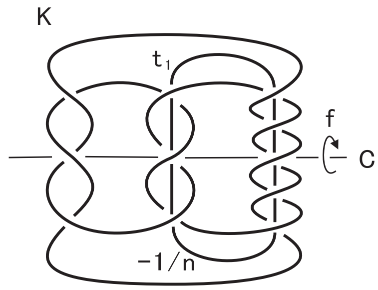

FIGURE 3.

Hence, $f|S^3-\mathrm{int}N(t_1)$ extends to an involution $\bar{f}$ of $(t_1; -1/n) \cong S^3$ about an axis $\overline{C}$ such that $\bar{f}(K_n) = K_n$ and $K_n \cap \overline{C} = \emptyset$. It follows that K_n has cyclic period 2. □ (Claim 2.1)

Claim 2.2. $(K_n; 1)$ *is a Seifert fiber space of type* $S^2(3, 5, |15n + 4|)$.

Proof. Let $(K \cup t_1;\ 1, -1/n)$ denote the manifold obtained by performing a surgery on the link $K \cup t_1$ with surgery slopes 1 for K and $-1/n$ for t_1. We will show that $(K \cup t_1;\ 1, -1/n)$ is a Seifert fiber space of type $S^2(3, 5, |15n + 4|)$.

To prove this we form the quotient by the involution $f : S^3 \to S^3$ to obtain the factor knot K_f, the branched knot c which is the image of C, and the arc τ_1 which is the image of t_1 and connects two points in c (Figure 4).

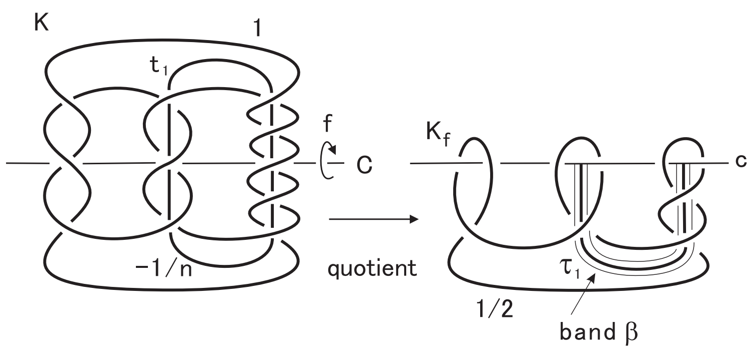

FIGURE 4.

As shown in Figure 4, the factor knot K_f is unknotted in $S^3/f \cong S^3$. Note that 1-surgery on K corresponds to 1/2-surgery on the factor knot K_f which is equivalent to (-2)-twisting along K_f because K_f is unknotted; see Figure 6. We denote the image of c after (-2)-twisting along K_f by c'. Note also that by the Montesinos trick ([19], [3]), $-1/n$-surgery on t_1 corresponds to $-1/n$-untangle surgery (i.e., a replacement of a 1/0-untangle by a $-1/n$-untangle) on c' along τ_1 as indicated in Figure 8. In order to correctly perform the untangle surgery, we keep track of the framing. This can be done by indicating a band β whose core is τ_1; see Figure 4. (For simplicity, we indicate the band β in only two places: just after taking the quotient by the involution f, and just before performing the untangle surgery.) By an isotopy as in Figures 6 and 7, we see that c' is the Montesinos knot given by the triple of rational tangles $(2/5, -3/4, 1/3)$. Denote the result of $-1/n$-untangle surgery on c' by c'_n (Figure 8). Then c'_n is the Montesinos knot given by the triple of rational tangles $(2/5, (11n+3)/(-15n-4), 1/3)$, and the branched covering space $(K \cup t_1;\ 1, -1/n)$ of S^3 branched along c'_n is a Seifert fiber space of type $S^2(3, 5, |15n+4|)$. Since the linking number of K and t_1 is zero, the 1-slope of K corresponds to the 1-slope of K_n, and hence $(K \cup t_1;\ 1, -1/n) \cong (K_n; 1)$. It follows that $(K_n; 1)$ is a Seifert fiber space of type $S^2(3, 5, |15n + 4|)$ as required. □ (Claim 2.2)

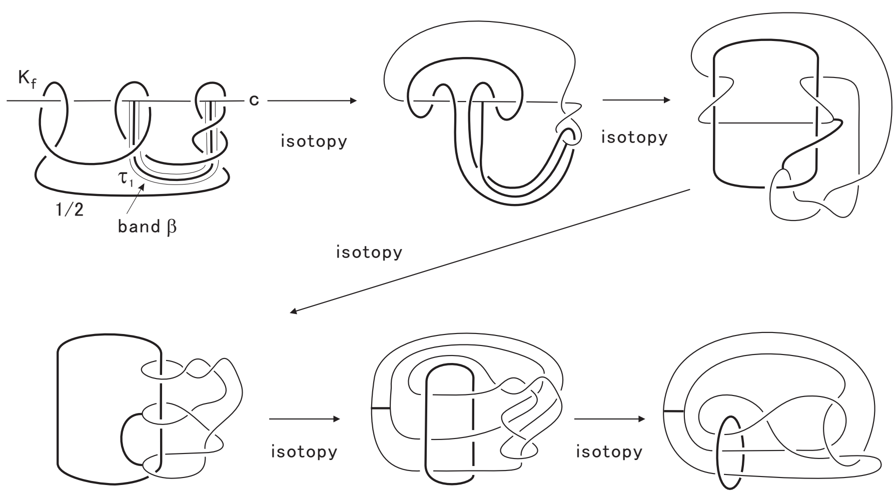

FIGURE 5.

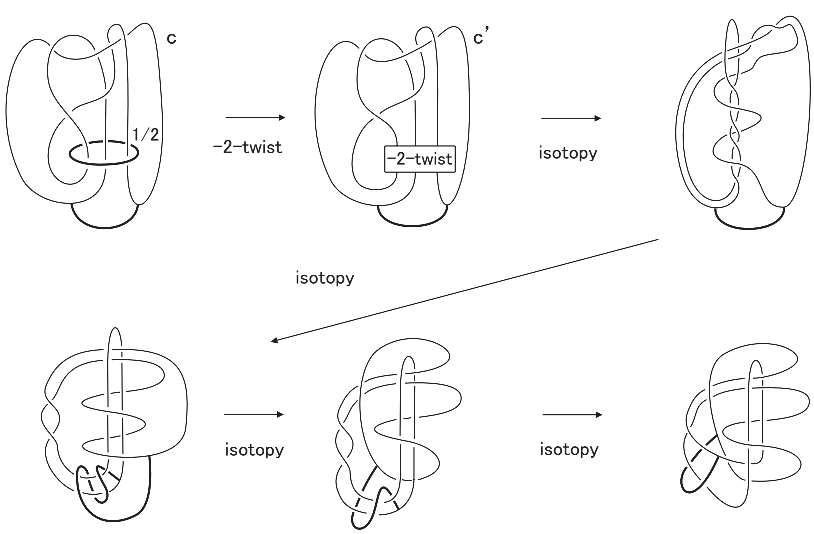

FIGURE 6. Continued from Figure 5.

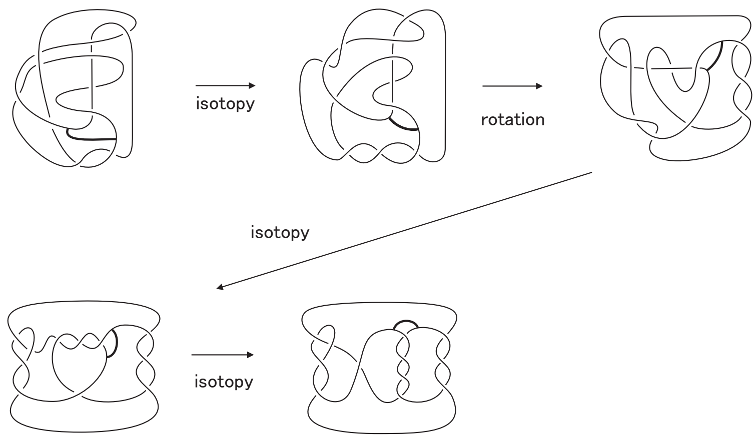

FIGURE 7. Continued from Figure 6.

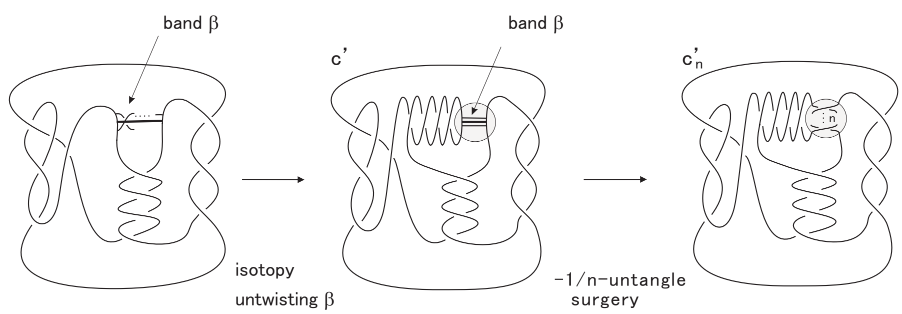

FIGURE 8. Continued from Figure 7.

Claim 2.3. *K_n is a hyperbolic knot.*

Proof. The knot K bounds an obvious Seifert surface S of genus one. Since t_1 can be isotoped off S, after doing n-twisting along t_1, S becomes a Seifert surface for K_n. By Claim 2.2, K_n is a nontrivial knot and thus $g(K_n)$, the genus of K_n, is equal to one.

Assume for a contradiction that K_n is a satellite knot. Then since $(K_n; 1)$ is atoroidal, K_n has a companion solid torus V whose core is a simple knot $\widehat{K_n}$ such that K_n is a 0 or 1-bridge braid in V ([16, Proposition 2.2(1)]). From Schubert's formula [23] ([6, Proposition 2.10]) we have $g(K_n) \geq wg(\widehat{K_n})$, where w denotes the winding number of K_n in V. Since $w \geq 2$ and $g(\widehat{K_n}) \geq 1$, we have $g(K_n) \geq 2$, a contradiction. If K_n is a torus knot, then since the genus is one, K_n is a $(\pm 2, 3)$-torus knot $T_{\pm 2,3}$. However, $(T_{2,3}; 1)$ (resp. $(T_{-2,3}; 1)$) is a Seifert fiber space of type $S^2(2, 3, 5)$ (resp. $S^2(2, 3, 7)$), contradicting Claim 2.2. It follows that K_n is a hyperbolic knot. □ (Claim 2.3)

Claim 2.4. *K_n is not strongly invertible.*

Proof. Recall that K_n has cyclic period 2 and that $(K_n; 1)$ is a Seifert fiber space of type $S^2(3, 5, |15n + 4|)$ (Claim 2.2). Since $|15n + 4| > 2$ and $|15n + 4| \neq 3,\ 5,$

if K_n is strongly invertible, then by [22, Theorem 1.7(1)], K_n is a torus knot or a cable of a torus knot. This contradicts K_n being hyperbolic (Claim 2.3). Therefore K_n is not strongly invertible. □ (Claim 2.4)

Claim 2.5. *The tunnel number of K_n is two.*

Proof. Let H be a handlebody in S^3 which is obtained by thickening the obvious genus one Seifert surface for K. Then $F = \partial H$ is a genus 2 Heegaard surface for S^3 which contains K. Since t_1 is a core of a handlebody H, H remains a handlebody after $-1/n$-surgery on t_1. It follows that K_n is embedded in a genus 2 Heegaard surface F. Then, by [20, Fact on p. 138] the tunnel number of K_n is less than or equal to 2. On the other hand, since a tunnel number one knot is strongly invertible ([21, Lemma 5]), Claim 2.4 implies that the tunnel number of K_n is two. □ (Claim 2.5)

Example 2. The second example is a variant of Example 1. Let us consider the trivial knot t_2 of Figure 9, instead of t_1 of Figure 1.

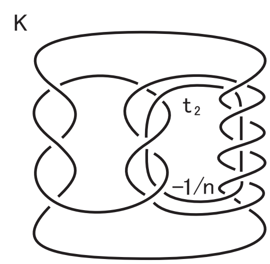

FIGURE 9.

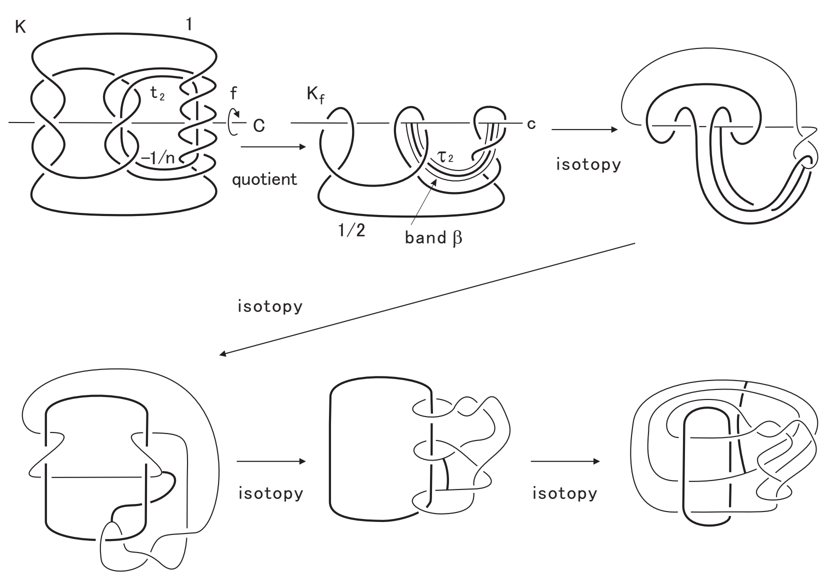

FIGURE 10.

Let K'_n be the knot obtained from K by doing n-twisting along t_2. Then the argument in the proof of Claim 2.2 shows that $(K'_n; 1)$ is a Seifert fiber space of type $S^2(3, 4, |12n + 5|)$; see Figures 10–13. The arguments in the proofs of Claims 2.1, 2.3, 2.4 and 2.5 show that the K'_n also enjoy the same properties as in Example 1, and that the Seifert fibered surgeries do not come from the primitive/Seifert-fibered construction.

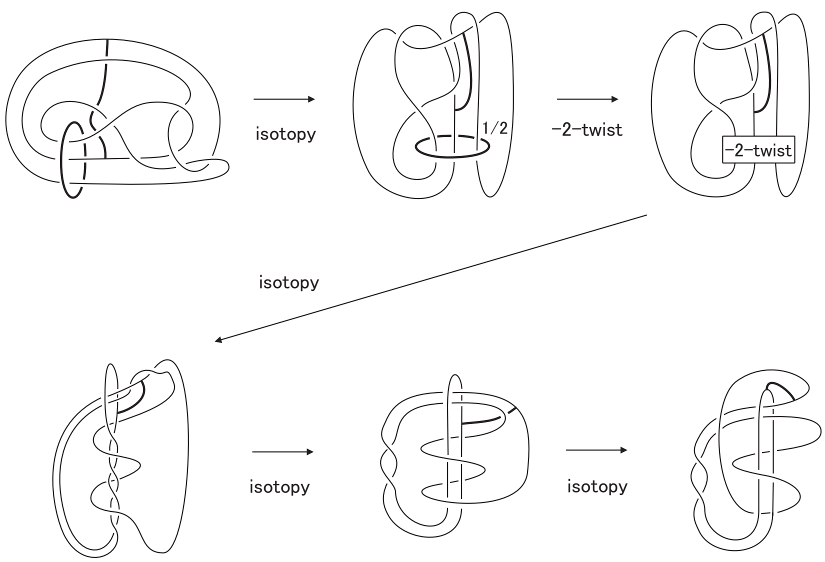

FIGURE 11. Continued from Figure 10.

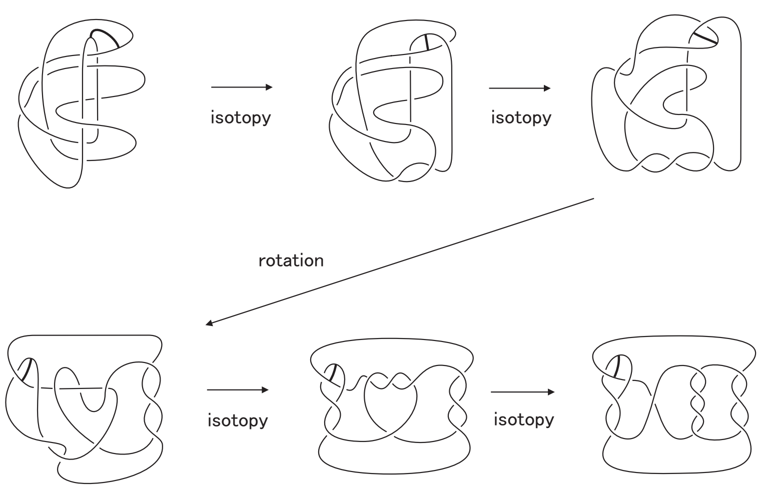

FIGURE 12. Continued from Figure 11.

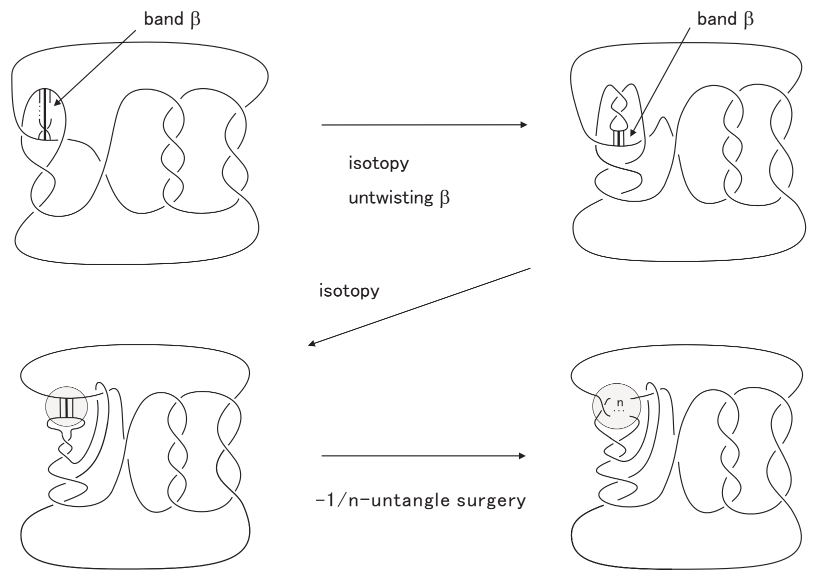

FIGURE 13. Continued from Figure 12.

3. REMARKS AND QUESTIONS

In [17] it has been conjectured that if $(K;r)$ is a Seifert fiber space, then it admits a Seifert fibration such that one of its fibers is unknotted in (the original) S^3. For our knots K_n (resp. K'_n), the trivial knot t_1^* which is the dual of t_1 (i.e., the core knot of $-1/n$-filling along t_1) (resp. t_2^* which is the dual of t_2) becomes an exceptional fiber of index $|15n+4|$ in $(K_n;1)$ (resp. an exceptional fiber of index $|12n+5|$ in $(K'_n;1)$). Thus the Dehn surgeries described in Examples 1 and 2 satisfy the conjecture. (Song's example mentioned in the Introduction also satisfies the conjecture.)

We also mention a geometric aspect of Seifert-fibered surgeries on hyperbolic knots. It was observed in [17, Section 7] that short closed geodesics in hyperbolic knot complements are often unknotted in S^3 and become Seifert fibers in the resulting Seifert fiber spaces after Dehn surgery. An experiment via Weeks' computer program SnapPea [24] suggests the table below, where K is the $(-3,3,5)$-pretzel knot, and t_1, t_2 are trivial knots described in Figures 1 and 9. Recall that $(K;1)$ is a Seifert fiber space of type $S^2(3,4,5)$.

	$S^3 - K$	S^3	$(K;1)$
t_1	third shortest geodesic	unknot	fiber of index 4
t_2	shortest geodesic	unknot	fiber of index 5

The second shortest geodesic is unknotted in S^3, but it does not become a fiber in $(K;1)$. In fact, it is hyperbolic in $(K;1)$.

We conclude this paper with some questions. Although the knots given in Examples 1 and 2 cannot be primitive/Seifert-fibered for any genus two Heegaard surface, they are still embedded in a genus two Heegaard surface for S^3. We would like to ask:

Question 3.1. *If $(K;r)$ is a Seifert fiber space, then is K embedded in a genus two Heegaard surface for S^3?*

In particular,

Question 3.2. *If $(K;r)$ is a Seifert fiber space, then is the tunnel number of K at most 2?*

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